

Second International Olympiad, 1960

1960/1.

Determine all three-digit numbers N having the property that N is divisible by 11, and $N/11$ is equal to the sum of the squares of the digits of N .

1960/2.

For what values of the variable x does the following inequality hold:

$$\frac{4x^2}{(1 - \sqrt{1 + 2x})^2} < 2x + 9?$$

1960/3.

In a given right triangle ABC , the hypotenuse BC , of length a , is divided into n equal parts (n an odd integer). Let α be the acute angle subtending, from A , that segment which contains the midpoint of the hypotenuse. Let h be the length of the altitude to the hypotenuse of the triangle. Prove:

$$\tan \alpha = \frac{4nh}{(n^2 - 1)a}.$$

1960/4.

Construct triangle ABC , given h_a, h_b (the altitudes from A and B) and m_a , the median from vertex A .

1960/5.

Consider the cube $ABCD A' B' C' D'$ (with face $ABCD$ directly above face $A' B' C' D'$).

- Find the locus of the midpoints of segments XY , where X is any point of AC and Y is any point of $B'D'$.
- Find the locus of points Z which lie on the segments XY of part (a) with $ZY = 2XZ$.

1960/6.

Consider a cone of revolution with an inscribed sphere tangent to the base of the cone. A cylinder is circumscribed about this sphere so that one of its bases lies in the base of the cone. Let V_1 be the volume of the cone and V_2 the volume of the cylinder.

- Prove that $V_1 \neq V_2$.
- Find the smallest number k for which $V_1 = kV_2$, for this case, construct the angle subtended by a diameter of the base of the cone at the vertex of the cone.

1960/7.

An isosceles trapezoid with bases a and c and altitude h is given.

- (a) On the axis of symmetry of this trapezoid, find all points P such that both legs of the trapezoid subtend right angles at P .
- (b) Calculate the distance of P from either base.
- (c) Determine under what conditions such points P actually exist. (Discuss various cases that might arise.)

Third International Olympiad, 1961

1961/1.

Solve the system of equations:

$$\begin{aligned}x + y + z &= a \\x^2 + y^2 + z^2 &= b^2 \\xy &= z^2\end{aligned}$$

where a and b are constants. Give the conditions that a and b must satisfy so that x, y, z (the solutions of the system) are distinct positive numbers.

1961/2.

Let a, b, c be the sides of a triangle, and T its area. Prove: $a^2 + b^2 + c^2 \geq 4\sqrt{3}T$. In what case does equality hold?

1961/3.

Solve the equation $\cos^n x - \sin^n x = 1$, where n is a natural number.

1961/4.

Consider triangle $P_1P_2P_3$ and a point P within the triangle. Lines P_1P, P_2P, P_3P intersect the opposite sides in points Q_1, Q_2, Q_3 respectively. Prove that, of the numbers

$$\frac{P_1P}{PQ_1}, \frac{P_2P}{PQ_2}, \frac{P_3P}{PQ_3}$$

at least one is ≤ 2 and at least one is ≥ 2 .

1961/5.

Construct triangle ABC if $AC = b, AB = c$ and $\angle AMB = \omega$, where M is the midpoint of segment BC and $\omega < 90^\circ$. Prove that a solution exists if and only if

$$b \tan \frac{\omega}{2} \leq c < b.$$

In what case does the equality hold?

1961/6.

Consider a plane ε and three non-collinear points A, B, C on the same side of ε ; suppose the plane determined by these three points is not parallel to ε . In plane α take three arbitrary points A', B', C' . Let L, M, N be the midpoints of segments AA', BB', CC' ; let G be the centroid of triangle LMN . (We will not consider positions of the points A', B', C' such that the points L, M, N do not form a triangle.) What is the locus of point G as A', B', C' range independently over the plane ε ?

Fourth International Olympiad, 1962

1962/1.

Find the smallest natural number n which has the following properties:

- (a) Its decimal representation has 6 as the last digit.
- (b) If the last digit 6 is erased and placed in front of the remaining digits, the resulting number is four times as large as the original number n .

1962/2.

Determine all real numbers x which satisfy the inequality:

$$\sqrt{3-x} - \sqrt{x+1} > \frac{1}{2}.$$

1962/3.

Consider the cube $ABCD A' B' C' D'$ ($ABCD$ and $A' B' C' D'$ are the upper and lower bases, respectively, and edges AA', BB', CC', DD' are parallel). The point X moves at constant speed along the perimeter of the square $ABCD$ in the direction $ABCD A$, and the point Y moves at the same rate along the perimeter of the square $B' C' C B B'$ in the direction $B' C' C B B'$. Points X and Y begin their motion at the same instant from the starting positions A and B' , respectively. Determine and draw the locus of the midpoints of the segments XY .

1962/4.

Solve the equation $\cos^2 x + \cos^2 2x + \cos^2 3x = 1$.

1962/5.

On the circle K there are given three distinct points A, B, C . Construct (using only straightedge and compasses) a fourth point D on K such that a circle can be inscribed in the quadrilateral thus obtained.

1962/6.

Consider an isosceles triangle. Let r be the radius of its circumscribed circle and ρ the radius of its inscribed circle. Prove that the distance d between the centers of these two circles is

$$d = \sqrt{r(r - 2\rho)}.$$

1962/7.

The tetrahedron $SABC$ has the following property: there exist five spheres, each tangent to the edges $SA, SB, SC, BCCA, AB$, or to their extensions.

(a) Prove that the tetrahedron $SABC$ is regular.

(b) Prove conversely that for every regular tetrahedron five such spheres exist.

Fifth International Olympiad, 1963

1963/1.

Find all real roots of the equation

$$\sqrt{x^2 - p} + 2\sqrt{x^2 - 1} = x,$$

where p is a real parameter.

1963/2.

Point A and segment BC are given. Determine the locus of points in space which are vertices of right angles with one side passing through A , and the other side intersecting the segment BC .

1963/3.

In an n -gon all of whose interior angles are equal, the lengths of consecutive sides satisfy the relation

$$a_1 \geq a_2 \geq \cdots \geq a_n.$$

Prove that $a_1 = a_2 = \cdots = a_n$.

1963/4.

Find all solutions x_1, x_2, x_3, x_4, x_5 of the system

$$x_5 + x_2 = yx_1$$

$$x_1 + x_3 = yx_2$$

$$x_2 + x_4 = yx_3$$

$$x_3 + x_5 = yx_4$$

$$x_4 + x_1 = yx_5,$$

where y is a parameter.

1963/5.

Prove that $\cos \frac{\pi}{7} - \cos \frac{2\pi}{7} + \cos \frac{3\pi}{7} = \frac{1}{2}$.

1963/6.

Five students, A, B, C, D, E , took part in a contest. One prediction was that the contestants would finish in the order $ABCDE$. This prediction was very poor. In fact no contestant finished in the position predicted, and no two contestants predicted to finish consecutively actually did so. A second prediction had the contestants finishing in the order $DAECB$. This prediction was better. Exactly two of the contestants finished in the places predicted, and two disjoint pairs of students predicted to finish consecutively actually did so. Determine the order in which the contestants finished.

Sixth International Olympiad, 1964

1964/1.

- (a) Find all positive integers n for which $2^n - 1$ is divisible by 7.
(b) Prove that there is no positive integer n for which $2^n + 1$ is divisible by 7.

1964/2.

Suppose a, b, c are the sides of a triangle. Prove that

$$a^2(b + c - a) + b^2(c + a - b) + c^2(a + b - c) \leq 3abc.$$

1964/3.

A circle is inscribed in triangle ABC with sides a, b, c . Tangents to the circle parallel to the sides of the triangle are constructed. Each of these tangents cuts off a triangle from $\triangle ABC$. In each of these triangles, a circle is inscribed. Find the sum of the areas of all four inscribed circles (in terms of a, b, c).

1964/4.

Seventeen people correspond by mail with one another - each one with all the rest. In their letters only three different topics are discussed. Each pair of correspondents deals with only one of these topics. Prove that there are at least three people who write to each other about the same topic.

1964/5.

Suppose five points in a plane are situated so that no two of the straight lines joining them are parallel, perpendicular, or coincident. From each point perpendiculars are drawn to all the lines joining the other four points. Determine the maximum number of intersections that these perpendiculars can have.

1964/6.

In tetrahedron $ABCD$, vertex D is connected with D_0 the centroid of $\triangle ABC$. Lines parallel to DD_0 are drawn through A, B and C . These lines intersect the planes BCD, CAD and ABD in points A_1, B_1 and C_1 , respectively. Prove that the volume of $ABCD$ is one third the volume of $A_1B_1C_1D_0$. Is the result true if point D_0 is selected anywhere within $\triangle ABC$?

Seventh Internatioaal Olympiad, 1965

1965/1.

Determine all values x in the interval $0 \leq x \leq 2\pi$ which satisfy the inequality

$$2 \cos x \leq \left| \sqrt{1 + \sin 2x} - \sqrt{1 - \sin 2x} \right| \leq \sqrt{2}.$$

1965/2.

Consider the system of equations

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}$$

with unknowns x_1, x_2, x_3 . The coefficients satisfy the conditions:

- (a) a_{11}, a_{22}, a_{33} are positive numbers;
- (b) the remaining coefficients are negative numbers;
- (c) in each equation, the sum of the coefficients is positive.

Prove that the given system has only the solution $x_1 = x_2 = x_3 = 0$.

1965/3.

Given the tetrahedron $ABCD$ whose edges AB and CD have lengths a and b respectively. The distance between the skew lines AB and CD is d , and the angle between them is ω . Tetrahedron $ABCD$ is divided into two solids by plane ε , parallel to lines AB and CD . The ratio of the distances of ε from AB and CD is equal to k . Compute the ratio of the volumes of the two solids obtained.

1965/4.

Find all sets of four real numbers x_1, x_2, x_3, x_4 such that the sum of any one and the product of the other three is equal to 2.

1965/5.

Consider $\triangle OAB$ with acute angle AOB . Through a point $M \neq O$ perpendiculars are drawn to OA and OB , the feet of which are P and Q respectively. The point of intersection of the altitudes of $\triangle OPQ$ is H . What is the locus of H if M is permitted to range over (a) the side AB , (b) the interior of $\triangle OAB$?

1965/6.

In a plane a set of n points ($n \geq 3$) is given. Each pair of points is connected by a segment. Let d be the length of the longest of these segments. We define a diameter of the set to be any connecting segment of length d . Prove that the number of diameters of the given set is at most n .

1966/6.

In the interior of sides BC, CA, AB of triangle ABC , any points K, L, M , respectively, are selected. Prove that the area of at least one of the triangles AML, BKM, CLK is less than or equal to one quarter of the area of triangle ABC .

Tenth International Olympiad, 1968

1968/1.

Prove that there is one and only one triangle whose side lengths are consecutive integers, and one of whose angles is twice as large as another.

1968/2.

Find all natural numbers x such that the product of their digits (in decimal notation) is equal to $x^2 - 10x - 22$.

1968/3.

Consider the system of equations

$$\begin{aligned} ax_1^2 + bx_1 + c &= x_2 \\ ax_2^2 + bx_2 + c &= x_3 \\ &\dots \\ ax_{n-1}^2 + bx_{n-1} + c &= x_n \\ ax_n^2 + bx_n + c &= x_1, \end{aligned}$$

with unknowns x_1, x_2, \dots, x_n , where a, b, c are real and $a \neq 0$. Let $\Delta = (b-1)^2 - 4ac$. Prove that for this system

- (a) if $\Delta < 0$, there is no solution,
- (b) if $\Delta = 0$, there is exactly one solution,
- (c) if $\Delta > 0$, there is more than one solution.

1968/4.

Prove that in every tetrahedron there is a vertex such that the three edges meeting there have lengths which are the sides of a triangle.

1968/5.

Let f be a real-valued function defined for all real numbers x such that, for some positive constant a , the equation

$$f(x+a) = \frac{1}{2} + \sqrt{f(x) - [f(x)]^2}$$

holds for all x .

- (a) Prove that the function f is periodic (i.e., there exists a positive number b such that $f(x+b) = f(x)$ for all x).
- (b) For $a = 1$, give an example of a non-constant function with the required properties.

1968/6.

For every natural number n , evaluate the sum

$$\sum_{k=0}^{\infty} \left[\frac{n+2^k}{2^{k+1}} \right] = \left[\frac{n+1}{2} \right] + \left[\frac{n+2}{4} \right] + \cdots + \left[\frac{n+2^k}{2^{k+1}} \right] + \cdots$$

(The symbol $[x]$ denotes the greatest integer not exceeding x .)

Eleventh International Olympiad, 1969

1969/1.

Prove that there are infinitely many natural numbers a with the following property: the number $z = n^4 + a$ is not prime for any natural number n .

1969/2.

Let a_1, a_2, \dots, a_n be real constants, x a real variable, and

$$f(x) = \cos(a_1 + x) + \frac{1}{2} \cos(a_2 + x) + \frac{1}{4} \cos(a_3 + x) \\ + \dots + \frac{1}{2^{n-1}} \cos(a_n + x).$$

Given that $f(x_1) = f(x_2) = 0$, prove that $x_2 - x_1 = m\pi$ for some integer m .

1969/3.

For each value of $k = 1, 2, 3, 4, 5$, find necessary and sufficient conditions on the number $a > 0$ so that there exists a tetrahedron with k edges of length a , and the remaining $6 - k$ edges of length 1.

1969/4.

A semicircular arc γ is drawn on AB as diameter. C is a point on γ other than A and B , and D is the foot of the perpendicular from C to AB . We consider three circles, $\gamma_1, \gamma_2, \gamma_3$, all tangent to the line AB . Of these, γ_1 is inscribed in $\triangle ABC$, while γ_2 and γ_3 are both tangent to CD and to γ , one on each side of CD . Prove that γ_1, γ_2 and γ_3 have a second tangent in common.

1969/5.

Given $n > 4$ points in the plane such that no three are collinear. Prove that there are at least $\binom{n-3}{2}$ convex quadrilaterals whose vertices are four of the given points.

1969/6.

Prove that for all real numbers $x_1, x_2, y_1, y_2, z_1, z_2$, with $x_1 > 0, x_2 > 0, x_1y_1 - z_1^2 > 0, x_2y_2 - z_2^2 > 0$, the inequality

$$\frac{8}{(x_1 + x_2)(y_1 + y_2) - (z_1 + z_2)^2} \leq \frac{1}{x_1y_1 - z_1^2} + \frac{1}{x_2y_2 - z_2^2}$$

is satisfied. Give necessary and sufficient conditions for equality.

Twelfth International Olympiad, 1970

1970/1.

Let M be a point on the side AB of $\triangle ABC$. Let r_1, r_2 and r be the radii of the inscribed circles of triangles AMC, BMC and ABC . Let q_1, q_2 and q be the radii of the escribed circles of the same triangles that lie in the angle ACB . Prove that

$$\frac{r_1}{q_1} \cdot \frac{r_2}{q_2} = \frac{r}{q}.$$

1970/2.

Let a, b and n be integers greater than 1, and let a and b be the bases of two number systems. A_{n-1} and A_n are numbers in the system with base a , and B_{n-1} and B_n are numbers in the system with base b ; these are related as follows:

$$\begin{aligned} A_n &= x_n x_{n-1} \cdots x_0, & A_{n-1} &= x_{n-1} x_{n-2} \cdots x_0, \\ B_n &= x_n x_{n-1} \cdots x_0, & B_{n-1} &= x_{n-1} x_{n-2} \cdots x_0, \\ x_n &\neq 0, & x_{n-1} &\neq 0. \end{aligned}$$

Prove:

$$\frac{A_{n-1}}{A_n} < \frac{B_{n-1}}{B_n} \text{ if and only if } a > b.$$

1970/3.

The real numbers $a_0, a_1, \dots, a_n, \dots$ satisfy the condition:

$$1 = a_0 \leq a_1 \leq a_2 \leq \cdots \leq a_n \leq \cdots.$$

The numbers $b_1, b_2, \dots, b_n, \dots$ are defined by

$$b_n = \sum_{k=1}^n \left(1 - \frac{a_{k-1}}{a_k}\right) \frac{1}{\sqrt{a_k}}.$$

(a) Prove that $0 \leq b_n < 2$ for all n .

(b) Given c with $0 \leq c < 2$, prove that there exist numbers a_0, a_1, \dots with the above properties such that $b_n > c$ for large enough n .

1970/4.

Find the set of all positive integers n with the property that the set $\{n, n+1, n+2, n+3, n+4, n+5\}$ can be partitioned into two sets such that the product of the numbers in one set equals the product of the numbers in the other set.

1970/5.

In the tetrahedron $ABCD$, angle BDC is a right angle. Suppose that the foot H of the perpendicular from D to the plane ABC is the intersection of the altitudes of $\triangle ABC$. Prove that

$$(AB + BC + CA)^2 \leq 6(AD^2 + BD^2 + CD^2).$$

For what tetrahedra does equality hold?

1970/6.

In a plane there are 100 points, no three of which are collinear. Consider all possible triangles having these points as vertices. Prove that no more than 70% of these triangles are acute-angled.

Thirteenth International Olympiad, 1971

1971/1.

Prove that the following assertion is true for $n = 3$ and $n = 5$, and that it is false for every other natural number $n > 2$:

If a_1, a_2, \dots, a_n are arbitrary real numbers, then

$$(a_1 - a_2)(a_1 - a_3) \cdots (a_1 - a_n) + (a_2 - a_1)(a_2 - a_3) \cdots (a_2 - a_n) \\ + \cdots + (a_n - a_1)(a_n - a_2) \cdots (a_n - a_{n-1}) \geq 0$$

1971/2.

Consider a convex polyhedron P_1 with nine vertices A_1A_2, \dots, A_9 ; let P_i be the polyhedron obtained from P_1 by a translation that moves vertex A_1 to A_i ($i = 2, 3, \dots, 9$). Prove that at least two of the polyhedra P_1, P_2, \dots, P_9 have an interior point in common.

1971/3.

Prove that the set of integers of the form $2^k - 3$ ($k = 2, 3, \dots$) contains an infinite subset in which every two members are relatively prime.

1971/4.

All the faces of tetrahedron $ABCD$ are acute-angled triangles. We consider all closed polygonal paths of the form $XYZTX$ defined as follows: X is a point on edge AB distinct from A and B ; similarly, Y, Z, T are interior points of edges BC, CD, DA , respectively. Prove:

(a) If $\angle DAB + \angle BCD \neq \angle CDA + \angle ABC$, then among the polygonal paths, there is none of minimal length.

(b) If $\angle DAB + \angle BCD = \angle CDA + \angle ABC$, then there are infinitely many shortest polygonal paths, their common length being $2AC \sin(\alpha/2)$, where $\alpha = \angle BAC + \angle CAD + \angle DAB$.

1971/5.

Prove that for every natural number m , there exists a finite set S of points in a plane with the following property: For every point A in S , there are exactly m points in S which are at unit distance from A .

1971/6.

Let $A = (a_{ij})(i, j = 1, 2, \dots, n)$ be a square matrix whose elements are non-negative integers. Suppose that whenever an element $a_{ij} = 0$, the sum of the elements in the i th row and the j th column is $\geq n$. Prove that the sum of all the elements of the matrix is $\geq n^2/2$.

Fourteenth International Olympiad, 1972

1972/1.

Prove that from a set of ten distinct two-digit numbers (in the decimal system), it is possible to select two disjoint subsets whose members have the same sum.

1972/2.

Prove that if $n \geq 4$, every quadrilateral that can be inscribed in a circle can be dissected into n quadrilaterals each of which is inscribable in a circle.

1972/3.

Let m and n be arbitrary non-negative integers. Prove that

$$\frac{(2m)!(2n)!}{m!n!(m+n)!}$$

is an integer. ($0! = 1$.)

1972/4.

Find all solutions $(x_1, x_2, x_3, x_4, x_5)$ of the system of inequalities

$$\begin{aligned}(x_1^2 - x_3x_5)(x_2^2 - x_3x_5) &\leq 0 \\(x_2^2 - x_4x_1)(x_3^2 - x_4x_1) &\leq 0 \\(x_3^2 - x_5x_2)(x_4^2 - x_5x_2) &\leq 0 \\(x_4^2 - x_1x_3)(x_5^2 - x_1x_3) &\leq 0 \\(x_5^2 - x_2x_4)(x_1^2 - x_2x_4) &\leq 0\end{aligned}$$

where x_1, x_2, x_3, x_4, x_5 are positive real numbers.

1972/5.

Let f and g be real-valued functions defined for all real values of x and y , and satisfying the equation

$$f(x+y) + f(x-y) = 2f(x)g(y)$$

for all x, y . Prove that if $f(x)$ is not identically zero, and if $|f(x)| \leq 1$ for all x , then $|g(y)| \leq 1$ for all y .

1972/6.

Given four distinct parallel planes, prove that there exists a regular tetrahedron with a vertex on each plane.

Sixteenth International Olympiad, 1974

1974/1.

Three players A, B and C play the following game: On each of three cards an integer is written. These three numbers p, q, r satisfy $0 < p < q < r$. The three cards are shuffled and one is dealt to each player. Each then receives the number of counters indicated by the card he holds. Then the cards are shuffled again; the counters remain with the players.

This process (shuffling, dealing, giving out counters) takes place for at least two rounds. After the last round, A has 20 counters in all, B has 10 and C has 9. At the last round B received r counters. Who received q counters on the first round?

1974/2.

In the triangle ABC , prove that there is a point D on side AB such that CD is the geometric mean of AD and DB if and only if

$$\sin A \sin B \leq \sin^2 \frac{C}{2}.$$

1974/3.

Prove that the number $\sum_{k=0}^n \binom{2n+1}{2k+1} 2^{3k}$ is not divisible by 5 for any integer $n \geq 0$.

1974/4.

Consider decompositions of an 8×8 chessboard into p non-overlapping rectangles subject to the following conditions:

- (i) Each rectangle has as many white squares as black squares.
- (ii) If a_i is the number of white squares in the i -th rectangle, then $a_1 < a_2 < \dots < a_p$. Find the maximum value of p for which such a decomposition is possible. For this value of p , determine all possible sequences a_1, a_2, \dots, a_p .

1974/5.

Determine all possible values of

$$S = \frac{a}{a+b+d} + \frac{b}{a+b+c} + \frac{c}{b+c+d} + \frac{d}{a+c+d}$$

where a, b, c, d are arbitrary positive numbers.

1974/6.

Let P be a non-constant polynomial with integer coefficients. If $n(P)$ is the number of distinct integers k such that $(P(k))^2 = 1$, prove that $n(P) - \deg(P) \leq 2$, where $\deg(P)$ denotes the degree of the polynomial P .

Seventeenth International Olympiad, 1975

1975/1.

Let x_i, y_i ($i = 1, 2, \dots, n$) be real numbers such that

$$x_1 \geq x_2 \geq \dots \geq x_n \text{ and } y_1 \geq y_2 \geq \dots \geq y_n.$$

Prove that, if z_1, z_2, \dots, z_n is any permutation of y_1, y_2, \dots, y_n , then

$$\sum_{i=1}^n (x_i - y_i)^2 \leq \sum_{i=1}^n (x_i - z_i)^2.$$

1975/2.

Let a_1, a_2, a_3, \dots be an infinite increasing sequence of positive integers. Prove that for every $p \geq 1$ there are infinitely many a_m which can be written in the form

$$a_m = xa_p + ya_q$$

with x, y positive integers and $q > p$.

1975/3.

On the sides of an arbitrary triangle ABC , triangles ABR, BCP, CAQ are constructed externally with $\angle CBP = \angle CAQ = 45^\circ$, $\angle BCP = \angle ACQ = 30^\circ$, $\angle ABR = \angle BAR = 15^\circ$. Prove that $\angle QRP = 90^\circ$ and $QR = RP$.

1975/4.

When 4444^{4444} is written in decimal notation, the sum of its digits is A . Let B be the sum of the digits of A . Find the sum of the digits of B . (A and B are written in decimal notation.)

1975/5.

Determine, with proof, whether or not one can find 1975 points on the circumference of a circle with unit radius such that the distance between any two of them is a rational number.

1975/6.

Find all polynomials P , in two variables, with the following properties:

(i) for a positive integer n and all real t, x, y

$$P(tx, ty) = t^n P(x, y)$$

(that is, P is homogeneous of degree n),

(ii) for all real a, b, c ,

$$P(b + c, a) + P(c + a, b) + P(a + b, c) = 0,$$

(iii) $P(1, 0) = 1$.

Nineteenth International Mathematical Olympiad, 1977

1977/1.

Equilateral triangles ABK, BCL, CDM, DAN are constructed inside the square $ABCD$. Prove that the midpoints of the four segments KL, LM, MN, NK and the midpoints of the eight segments $AKBK, BL, CL, CM, DM, DN, AN$ are the twelve vertices of a regular dodecagon.

1977/2.

In a finite sequence of real numbers the sum of any seven successive terms is negative, and the sum of any eleven successive terms is positive. Determine the maximum number of terms in the sequence.

1977/3.

Let n be a given integer > 2 , and let V_n be the set of integers $1 + kn$, where $k = 1, 2, \dots$. A number $m \in V_n$ is called *indecomposable* in V_n if there do not exist numbers $p, q \in V_n$ such that $pq = m$. Prove that there exists a number $r \in V_n$ that can be expressed as the product of elements indecomposable in V_n in more than one way. (Products which differ only in the order of their factors will be considered the same.)

1977/4.

Four real constants a, b, A, B are given, and

$$f(\theta) = 1 - a \cos \theta - b \sin \theta - A \cos 2\theta - B \sin 2\theta.$$

Prove that if $f(\theta) \geq 0$ for all real θ , then

$$a^2 + b^2 \leq 2 \text{ and } A^2 + B^2 \leq 1.$$

1977/5.

Let a and b be positive integers. When $a^2 + b^2$ is divided by $a + b$, the quotient is q and the remainder is r . Find all pairs (a, b) such that $q^2 + r = 1977$.

1977/6.

Let $f(n)$ be a function defined on the set of all positive integers and having all its values in the same set. Prove that if

$$f(n+1) > f(f(n))$$

for each positive integer n , then

$$f(n) = n \text{ for each } n.$$

Twentieth International Olympiad, 1978

1978/1. m and n are natural numbers with $1 \leq m < n$. In their decimal representations, the last three digits of 1978^m are equal, respectively, to the last three digits of 1978^n . Find m and n such that $m + n$ has its least value.

1978/2. P is a given point inside a given sphere. Three mutually perpendicular rays from P intersect the sphere at points U, V , and W ; Q denotes the vertex diagonally opposite to P in the parallelepiped determined by PU, PV , and PW . Find the locus of Q for all such triads of rays from P .

1978/3. The set of all positive integers is the union of two disjoint subsets $\{f(1), f(2), \dots, f(n), \dots\}, \{g(1), g(2), \dots, g(n), \dots\}$, where

$$f(1) < f(2) < \dots < f(n) < \dots,$$

$$g(1) < g(2) < \dots < g(n) < \dots,$$

and

$$g(n) = f(f(n)) + 1 \text{ for all } n \geq 1.$$

Determine $f(240)$.

1978/4. In triangle ABC , $AB = AC$. A circle is tangent internally to the circumcircle of triangle ABC and also to sides AB, AC at P, Q , respectively. Prove that the midpoint of segment PQ is the center of the incircle of triangle ABC .

1978/5. Let $\{a_k\} (k = 1, 2, 3, \dots, n, \dots)$ be a sequence of distinct positive integers. Prove that for all natural numbers n ,

$$\sum_{k=1}^n \frac{a_k}{k^2} \geq \sum_{k=1}^n \frac{1}{k}.$$

1978/6. An international society has its members from six different countries. The list of members contains 1978 names, numbered $1, 2, \dots, 1978$. Prove that there is at least one member whose number is the sum of the numbers of two members from his own country, or twice as large as the number of one member from his own country.

Twenty-third International Olympiad, 1982

1982/1. The function $f(n)$ is defined for all positive integers n and takes on non-negative integer values. Also, for all m, n

$$f(m+n) - f(m) - f(n) = 0 \text{ or } 1$$

$$f(2) = 0, f(3) > 0, \text{ and } f(9999) = 3333.$$

Determine $f(1982)$.

1982/2. A non-isosceles triangle $A_1A_2A_3$ is given with sides a_1, a_2, a_3 (a_i is the side opposite A_i). For all $i = 1, 2, 3$, M_i is the midpoint of side a_i , and T_i is the point where the incircle touches side a_i . Denote by S_i the reflection of T_i in the interior bisector of angle A_i . Prove that the lines M_1S_1, M_2S_2 , and M_3S_3 are concurrent.

1982/3. Consider the infinite sequences $\{x_n\}$ of positive real numbers with the following properties:

$$x_0 = 1, \text{ and for all } i \geq 0, x_{i+1} \leq x_i.$$

(a) Prove that for every such sequence, there is an $n \geq 1$ such that

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \dots + \frac{x_{n-1}^2}{x_n} \geq 3.999.$$

(b) Find such a sequence for which

$$\frac{x_0^2}{x_1} + \frac{x_1^2}{x_2} + \dots + \frac{x_{n-1}^2}{x_n} < 4.$$

1982/4. Prove that if n is a positive integer such that the equation

$$x^3 - 3xy^2 + y^3 = n$$

has a solution in integers (x, y) , then it has at least three such solutions.

Show that the equation has no solutions in integers when $n = 2891$.

1982/5. The diagonals AC and CE of the regular hexagon $ABCDEF$ are divided by the inner points M and N , respectively, so that

$$\frac{AM}{AC} = \frac{CN}{CE} = r.$$

Determine r if B, M , and N are collinear.

1982/6. Let S be a square with sides of length 100, and let L be a path within S which does not meet itself and which is composed of line segments $A_0A_1, A_1A_2, \dots, A_{n-1}A_n$ with $A_0 \neq A_n$. Suppose that for every point P of the boundary of S there is a point of L at a distance from P not greater than $1/2$. Prove that there are two points X and Y in L such that the distance between X and Y is not greater than 1, and the length of that part of L which lies between X and Y is not smaller than 198.

Twenty-fifth International Olympiad, 1984

1984/1. Prove that $0 \leq yz + zx + xy - 2xyz \leq 7/27$, where x, y and z are non-negative real numbers for which $x + y + z = 1$.

1984/2. Find one pair of positive integers a and b such that:

(i) $ab(a + b)$ is not divisible by 7;

(ii) $(a + b)^7 - a^7 - b^7$ is divisible by 7^7 .

Justify your answer.

1984/3. In the plane two different points O and A are given. For each point X of the plane, other than O , denote by $a(X)$ the measure of the angle between OA and OX in radians, counterclockwise from OA ($0 \leq a(X) < 2\pi$). Let $C(X)$ be the circle with center O and radius of length $OX + a(X)/OX$. Each point of the plane is colored by one of a finite number of colors. Prove that there exists a point Y for which $a(Y) > 0$ such that its color appears on the circumference of the circle $C(Y)$.

1984/4. Let $ABCD$ be a convex quadrilateral such that the line CD is a tangent to the circle on AB as diameter. Prove that the line AB is a tangent to the circle on CD as diameter if and only if the lines BC and AD are parallel.

1984/5. Let d be the sum of the lengths of all the diagonals of a plane convex polygon with n vertices ($n > 3$), and let p be its perimeter. Prove that

$$n - 3 < \frac{2d}{p} < \left[\frac{n}{2} \right] \left[\frac{n+1}{2} \right] - 2,$$

where $[x]$ denotes the greatest integer not exceeding x .

1984/6. Let a, b, c and d be odd integers such that $0 < a < b < c < d$ and $ad = bc$. Prove that if $a + d = 2^k$ and $b + c = 2^m$ for some integers k and m , then $a = 1$.

Twenty-sixth International Olympiad, 1985

1985/1. A circle has center on the side AB of the cyclic quadrilateral $ABCD$. The other three sides are tangent to the circle. Prove that $AD + BC = AB$.

1985/2. Let n and k be given relatively prime natural numbers, $k < n$. Each number in the set $M = \{1, 2, \dots, n - 1\}$ is colored either blue or white. It is given that

(i) for each $i \in M$, both i and $n - i$ have the same color;

(ii) for each $i \in M, i \neq k$, both i and $|i - k|$ have the same color. Prove that all numbers in M must have the same color.

1985/3. For any polynomial $P(x) = a_0 + a_1x + \dots + a_kx^k$ with integer coefficients, the number of coefficients which are odd is denoted by $w(P)$. For $i = 0, 1, \dots$, let $Q_i(x) = (1 + x)^i$. Prove that if i_1, i_2, \dots, i_n are integers such that $0 \leq i_1 < i_2 < \dots < i_n$, then

$$w(Q_{i_1} + Q_{i_2} + \dots + Q_{i_n}) \geq w(Q_{i_1}).$$

1985/4. Given a set M of 1985 distinct positive integers, none of which has a prime divisor greater than 26. Prove that M contains at least one subset of four distinct elements whose product is the fourth power of an integer.

1985/5. A circle with center O passes through the vertices A and C of triangle ABC and intersects the segments AB and BC again at distinct points K and N , respectively. The circumscribed circles of the triangles ABC and EBN intersect at exactly two distinct points B and M . Prove that angle OMB is a right angle.

1985/6. For every real number x_1 , construct the sequence x_1, x_2, \dots by setting

$$x_{n+1} = x_n \left(x_n + \frac{1}{n} \right) \text{ for each } n \geq 1.$$

Prove that there exists exactly one value of x_1 for which

$$0 < x_n < x_{n+1} < 1$$

for every n .

27th International Mathematical Olympiad

Warsaw, Poland

Day I

July 9, 1986

1. Let d be any positive integer not equal to 2, 5, or 13. Show that one can find distinct a, b in the set $\{2, 5, 13, d\}$ such that $ab - 1$ is not a perfect square.
2. A triangle $A_1A_2A_3$ and a point P_0 are given in the plane. We define $A_s = A_{s-3}$ for all $s \geq 4$. We construct a set of points P_1, P_2, P_3, \dots , such that P_{k+1} is the image of P_k under a rotation with center A_{k+1} through angle 120° clockwise (for $k = 0, 1, 2, \dots$). Prove that if $P_{1986} = P_0$, then the triangle $A_1A_2A_3$ is equilateral.
3. To each vertex of a regular pentagon an integer is assigned in such a way that the sum of all five numbers is positive. If three consecutive vertices are assigned the numbers x, y, z respectively and $y < 0$ then the following operation is allowed: the numbers x, y, z are replaced by $x + y, -y, z + y$ respectively. Such an operation is performed repeatedly as long as at least one of the five numbers is negative. Determine whether this procedure necessarily comes to an end after a finite number of steps.

27th International Mathematical Olympiad

Warsaw, Poland

Day II

July 10, 1986

4. Let A, B be adjacent vertices of a regular n -gon ($n \geq 5$) in the plane having center at O . A triangle XYZ , which is congruent to and initially coincides with OAB , moves in the plane in such a way that Y and Z each trace out the whole boundary of the polygon, X remaining inside the polygon. Find the locus of X .
5. Find all functions f , defined on the non-negative real numbers and taking non-negative real values, such that:
 - (i) $f(xf(y))f(y) = f(x + y)$ for all $x, y \geq 0$,
 - (ii) $f(2) = 0$,
 - (iii) $f(x) \neq 0$ for $0 \leq x < 2$.
6. One is given a finite set of points in the plane, each point having integer coordinates. Is it always possible to color some of the points in the set red and the remaining points white in such a way that for any straight line L parallel to either one of the coordinate axes the difference (in absolute value) between the numbers of white point and red points on L is not greater than 1?

30th International Mathematical Olympiad

Braunschweig, Germany

Day I

1. Prove that the set $\{1, 2, \dots, 1989\}$ can be expressed as the disjoint union of subsets A_i ($i = 1, 2, \dots, 117$) such that:
 - (i) Each A_i contains 17 elements;
 - (ii) The sum of all the elements in each A_i is the same.

2. In an acute-angled triangle ABC the internal bisector of angle A meets the circumcircle of the triangle again at A_1 . Points B_1 and C_1 are defined similarly. Let A_0 be the point of intersection of the line AA_1 with the external bisectors of angles B and C . Points B_0 and C_0 are defined similarly. Prove that:
 - (i) The area of the triangle $A_0B_0C_0$ is twice the area of the hexagon $AC_1BA_1CB_1$.
 - (ii) The area of the triangle $A_0B_0C_0$ is at least four times the area of the triangle ABC .

3. Let n and k be positive integers and let S be a set of n points in the plane such that
 - (i) No three points of S are collinear, and
 - (ii) For any point P of S there are at least k points of S equidistant from P .

Prove that:

$$k < \frac{1}{2} + \sqrt{2n}.$$

30th International Mathematical Olympiad

Braunschweig, Germany

Day II

4. Let $ABCD$ be a convex quadrilateral such that the sides AB , AD , BC satisfy $AB = AD + BC$. There exists a point P inside the quadrilateral at a distance h from the line CD such that $AP = h + AD$ and $BP = h + BC$. Show that:

$$\frac{1}{\sqrt{h}} \geq \frac{1}{\sqrt{AD}} + \frac{1}{\sqrt{BC}}.$$

5. Prove that for each positive integer n there exist n consecutive positive integers none of which is an integral power of a prime number.
6. A permutation (x_1, x_2, \dots, x_m) of the set $\{1, 2, \dots, 2n\}$, where n is a positive integer, is said to have property P if $|x_i - x_{i+1}| = n$ for at least one i in $\{1, 2, \dots, 2n - 1\}$. Show that, for each n , there are more permutations with property P than without.

31st International Mathematical Olympiad

Beijing, China

Day I

July 12, 1990

1. Chords AB and CD of a circle intersect at a point E inside the circle. Let M be an interior point of the segment EB . The tangent line at E to the circle through D , E , and M intersects the lines BC and AC at F and G , respectively. If

$$\frac{AM}{AB} = t,$$

find

$$\frac{EG}{EF}$$

in terms of t .

2. Let $n \geq 3$ and consider a set E of $2n - 1$ distinct points on a circle. Suppose that exactly k of these points are to be colored black. Such a coloring is “good” if there is at least one pair of black points such that the interior of one of the arcs between them contains exactly n points from E . Find the smallest value of k so that every such coloring of k points of E is good.
3. Determine all integers $n > 1$ such that

$$\frac{2^n + 1}{n^2}$$

is an integer.

31st International Mathematical Olympiad

Beijing, China

Day II

July 13, 1990

4. Let \mathbb{Q}^+ be the set of positive rational numbers. Construct a function $f : \mathbb{Q}^+ \rightarrow \mathbb{Q}^+$ such that

$$f(xf(y)) = \frac{f(x)}{y}$$

for all x, y in \mathbb{Q}^+ .

5. Given an initial integer $n_0 > 1$, two players, \mathcal{A} and \mathcal{B} , choose integers n_1, n_2, n_3, \dots alternately according to the following rules:

Knowing n_{2k} , \mathcal{A} chooses any integer n_{2k+1} such that

$$n_{2k} \leq n_{2k+1} \leq n_{2k}^2.$$

Knowing n_{2k+1} , \mathcal{B} chooses any integer n_{2k+2} such that

$$\frac{n_{2k+1}}{n_{2k+2}}$$

is a prime raised to a positive integer power.

Player \mathcal{A} wins the game by choosing the number 1990; player \mathcal{B} wins by choosing the number 1. For which n_0 does:

- (a) \mathcal{A} have a winning strategy?
 - (b) \mathcal{B} have a winning strategy?
 - (c) Neither player have a winning strategy?
6. Prove that there exists a convex 1990-gon with the following two properties:
- (a) All angles are equal.
 - (b) The lengths of the 1990 sides are the numbers $1^2, 2^2, 3^2, \dots, 1990^2$ in some order.

32nd International Mathematical Olympiad

First Day — July 17, 1991

Time Limit: $4\frac{1}{2}$ hours

1. Given a triangle ABC , let I be the center of its inscribed circle. The internal bisectors of the angles A, B, C meet the opposite sides in A', B', C' respectively. Prove that

$$\frac{1}{4} < \frac{AI \cdot BI \cdot CI}{AA' \cdot BB' \cdot CC'} \leq \frac{8}{27}.$$

2. Let $n > 6$ be an integer and a_1, a_2, \dots, a_k be all the natural numbers less than n and relatively prime to n . If

$$a_2 - a_1 = a_3 - a_2 = \dots = a_k - a_{k-1} > 0,$$

prove that n must be either a prime number or a power of 2.

3. Let $S = \{1, 2, 3, \dots, 280\}$. Find the smallest integer n such that each n -element subset of S contains five numbers which are pairwise relatively prime.

Second Day — July 18, 1991

Time Limit: $4\frac{1}{2}$ hours

1. Suppose G is a connected graph with k edges. Prove that it is possible to label the edges $1, 2, \dots, k$ in such a way that at each vertex which belongs to two or more edges, the greatest common divisor of the integers labeling those edges is equal to 1.

[A *graph* consists of a set of points, called *vertices*, together with a set of *edges* joining certain pairs of distinct vertices. Each pair of vertices u, v belongs to at most one edge. The graph G is *connected* if for each pair of distinct vertices x, y there is some sequence of vertices $x = v_0, v_1, v_2, \dots, v_m = y$ such that each pair v_i, v_{i+1} ($0 \leq i < m$) is joined by an edge of G .]

2. Let ABC be a triangle and P an interior point of ABC . Show that at least one of the angles $\angle PAB, \angle PBC, \angle PCA$ is less than or equal to 30° .

3. An infinite sequence x_0, x_1, x_2, \dots of real numbers is said to be *bounded* if there is a constant C such that $|x_i| \leq C$ for every $i \geq 0$.

Given any real number $a > 1$, construct a bounded infinite sequence x_0, x_1, x_2, \dots such that

$$|x_i - x_j| |i - j|^a \geq 1$$

for every pair of distinct nonnegative integers i, j .

**The 35th International Mathematical Olympiad (July 13-14,
1994, Hong Kong)**

1. Let m and n be positive integers. Let a_1, a_2, \dots, a_m be distinct elements of $\{1, 2, \dots, n\}$ such that whenever $a_i + a_j \leq n$ for some i, j , $1 \leq i \leq j \leq m$, there exists k , $1 \leq k \leq m$, with $a_i + a_j = a_k$. Prove that

$$\frac{a_1 + a_2 + \dots + a_m}{m} \geq \frac{n + 1}{2}.$$

2. ABC is an isosceles triangle with $AB = AC$. Suppose that

1. M is the midpoint of BC and O is the point on the line AM such that OB is perpendicular to AB ;
2. Q is an arbitrary point on the segment BC different from B and C ;
3. E lies on the line AB and F lies on the line AC such that E, Q, F are distinct and collinear.

Prove that OQ is perpendicular to EF if and only if $QE = QF$.

3. For any positive integer k , let $f(k)$ be the number of elements in the set $\{k + 1, k + 2, \dots, 2k\}$ whose base 2 representation has precisely three 1s.

- (a) Prove that, for each positive integer m , there exists at least one positive integer k such that $f(k) = m$.
- (b) Determine all positive integers m for which there exists exactly one k with $f(k) = m$.

4. Determine all ordered pairs (m, n) of positive integers such that

$$\frac{n^3 + 1}{mn - 1}$$

is an integer.

5. Let S be the set of real numbers strictly greater than -1 . Find all functions $f : S \rightarrow S$ satisfying the two conditions:

1. $f(x + f(y) + xf(y)) = y + f(x) + yf(x)$ for all x and y in S ;
2. $\frac{f(x)}{x}$ is strictly increasing on each of the intervals $-1 < x < 0$ and $0 < x$.

6. Show that there exists a set A of positive integers with the following property: For any infinite set S of primes there exist two positive integers $m \in A$ and $n \notin A$ each of which is a product of k distinct elements of S for some $k \geq 2$.

THE 36th IMO, CANADA, 1995

1 First Day, July 19, 1995

1. Let A, B, C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y . The line XY meets BC at the point Z . Let P be a point on the line XY different from Z . The line CP intersects the circle with diameter AC at the points C and M , and the line BP intersects the circle with diameter BD at the points B and N . Prove that the lines AM, DN , and XY are concurrent.

2. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

3. Determine all integers $n > 3$ such that there exist n points A_1, A_2, \dots, A_n in the plane, and real numbers r_1, r_2, \dots, r_n satisfying the following two conditions:

- (a) no three of the points A_1, A_2, \dots, A_n lie on a line;
- (b) for each triple $i, j, k (1 \leq i < j < k \leq n)$ the triangle $A_i A_j A_k$ has area equal to $r_i + r_j + r_k$.

2 Second Day, July 20, 1995

1. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \dots, x_{1995}$ satisfying the two conditions:

(a) $x_0 = x_{1995}$;

(b) $x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$ for each $i = 1, \dots, 1995$.

2. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$, and $\angle BCD = \angle EFA = 60^\circ$.

Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^\circ$. Prove that $AG + GB + GH + DH + HE \geq CF$.

3. Let p be an odd prime. Find the number of subsets A of $\{1, 2, \dots, 2p\}$ such that

(a) A has exactly p elements, and

(b) the sum of all the elements in A is divisible by p .

SOLUTIONS

3 First Day, July 19, 1995

1. Let A , B , C and D be four distinct points on a line, in that order. The circles with diameters AC and BD intersect at the points X and Y . The line XY meets BC at the point Z . Let P be a point on the line XY different from Z . The line CP intersects the circle with diameter AC at the points C and M , and the line BP intersects the circle with diameter BD at the points B and N . Prove that the lines AM , DN , and XY are concurrent.

Solution

Let the lines XY and DN meet at the point Q . Then triangles BPZ and QDZ are similar. Hence $\frac{ZQ}{ZD} = \frac{ZB}{ZP}$ or $ZQ = \frac{ZD \cdot ZB}{ZP}$. Since Z, D, B and P are fixed points, so is Q . By symmetry, AM also passes through Q .

Alternative Solution

Draw a line through A parallel to BP , cutting the line XY at the point W . Then triangles BPZ and AWZ are similar. Hence $\frac{ZP}{ZW} = \frac{ZB}{ZA}$. Since

$$ZA \cdot ZC = ZX \cdot ZY = ZB \cdot ZD,$$

we have $\frac{ZP}{ZW} = \frac{ZC}{ZD}$. It follows that triangles CPZ and DWZ are also similar, so that DW is parallel to CP . Since AM is perpendicular to CP , it is also perpendicular to DW . Similarly, DN is perpendicular to AW . Finally, XY is perpendicular to AD . Hence AM , DN and XY are concurrent at the orthocentre of triangle WAD .

Remark: This problem is very easy, and has many different solutions. They may be grouped into two types. The first solution, due independently to Sam Maltby and Johannes Notenboom, the leader of the team from the Netherlands, represents those in which only half of the diagram is considered. The second solution, due to Nazar Agakhanov, the leader of the team from Russia, represents those in which the whole diagram is considered. Other techniques used include Menelaus' Theorem and analytic geometry.

2. Let a, b and c be positive real numbers such that $abc = 1$. Prove that

$$\frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)} \geq \frac{3}{2}.$$

Solution

Let

$$S = \frac{1}{a^3(b+c)} + \frac{1}{b^3(a+c)} + \frac{1}{c^3(a+b)}$$

and define

$$x = \frac{1}{a}, \quad y = \frac{1}{b}, \quad z = \frac{1}{c} \quad \text{and} \quad T = x + y + z.$$

The positive real numbers x, y , and z satisfy $xyz = 1$. Then

$$\begin{aligned} \frac{1}{a^3(b+c)} &= \frac{x^3}{\frac{1}{y} + \frac{1}{z}} = \frac{x^3yz}{y+z} = \frac{x^3yz}{y+z} = \frac{x^2}{T-x} \\ &= \frac{T^2 - (T^2 - x^2)}{T-x} = \frac{T^2}{T-x} - T - x. \end{aligned}$$

This and the corresponding terms in y and z yield

$$T^2 \left(\frac{1}{T-x} + \frac{1}{T-y} + \frac{1}{T-z} \right) - 4T.$$

By the Arithmetic–Harmonic–Mean Inequality

$$S \geq T^2 \cdot \frac{9}{(T-x) + (T-y) + (T-z)} - 4T = \frac{9T}{2} - 4T = \frac{x+y+z}{2}.$$

By the Arithmetic–Geometric–Mean Inequality

$$S \geq \frac{3}{2} \sqrt[3]{xyz} = \frac{3}{2}.$$

Equality holds if and only if $x = y = z = 1$, which is equivalent to $a = b = c = 1$.

Alternative Solution

Let x, y, z and S be as in the first solution.

$$S = \frac{x^2}{y+z} + \frac{y^2}{z+x} + \frac{z^2}{x+y}.$$

By Cauchy's Inequality,

$$[(y+z) + (z+x) + (x+y)] S \geq (x+y+z)^2$$

or $S \geq \frac{x+y+z}{2}$. It follows from the Arithmetic-Geometric-Mean Inequality that

$$S \geq \frac{x+y+z}{3} \cdot \frac{3}{2} \geq \sqrt[3]{xyz} \cdot \frac{3}{2} = \frac{3}{2}.$$

Equality holds if and only if $x = y = z = 1$, which is equivalent to $a = b = c = 1$.

Remark: The first solution is due to the proposer, Nazar Agakhanov, the leader of the team from Russia. The second solution is due to Murray Klamkin. Several competitors applied Chebychev's Inequality directly to S (written in terms of x, y and z) followed by applications of the Root-Mean-Square-Arithmetic-Mean Inequality and Arithmetic-Geometric-Mean Inequality.

3. Determine all integers $n > 3$ such that there exist n points A_1, A_2, \dots, A_n in the plane, and real numbers r_1, r_2, \dots, r_n satisfying the following two conditions:
- (a) no three of the points A_1, A_2, \dots, A_n lie on a line;
 - (b) for each triple $i, j, k (1 \leq i < j < k \leq n)$ the triangle $A_i A_j A_k$ has area equal to $r_i + r_j + r_k$.

Solution

We claim that $n = 4$ is the only integer satisfying the conditions of the problem. For $n = 4$, let $A_1 A_2 A_3 A_4$ be a unit square and let $r_1 = r_2 = r_3 = r_4 = 1/6$. It remains to show that no solution exists for $n = 5$, which implies that there are no solutions for any $n \geq 5$.

Suppose to the contrary that there is a solution with $n = 5$. Denote the area of $\triangle A_i A_j A_k$ by $[ijk] = r_i + r_j + r_k, 1 \leq i < j < k \leq 5$. If $A_i A_j A_k A_\ell$ is convex, then $r_i + r_k = r_j + r_\ell$. This follows from $[ijk] + [kli] = [jkl] + [lij]$.

We cannot have $r_i = r_j$. If for instance $r_4 = r_5$, then $[124] = [125]$. If A_1 and A_2 are on the same side of $A_4 A_5$, then $A_1 A_2$ must be parallel to $A_4 A_5$. If they are on opposite sides, then $A_1 A_2$ must pass through the midpoint M of $A_4 A_5$. The same can be said about $A_2 A_3$ and $A_3 A_1$. Since A_1, A_2 and A_3 are not collinear, at most one of $A_1 A_2, A_2 A_3$ and $A_3 A_1$ can be parallel to $A_4 A_5$, and at most one can pass through M . This is a contradiction.

Consider the convex hull of A_1, A_2, A_3, A_4 and A_5 . We have three cases.

First, suppose that the convex hull is a pentagon $A_1 A_2 A_3 A_4 A_5$. Since $A_1 A_2 A_3 A_4$ and $A_1 A_2 A_3 A_5$ are convex, our observation yields $r_1 + r_3 = r_2 + r_4$ and $r_1 + r_3 = r_2 + r_5$. Hence $r_4 = r_5$, a contradiction.

Next, suppose that the convex hull is a quadrilateral $A_1 A_2 A_3 A_4$. We may assume that A_5 lies within $A_3 A_4 A_1$. Then $A_1 A_2 A_3 A_5$ is convex, and we have the same contradiction as before.

Finally, suppose that the convex hull is a triangle $A_1 A_2 A_3$. Since

$$[124] + [234] + [314] = [125] + [235] + [315],$$

we have $r_4 = r_5$, a contradiction.

Alternative Solution

We proceed as in the first solution up to $r_i + r_k = r_j + r_\ell$ if $A_i A_j A_k A_\ell$ is convex. Assume that $r_1 \geq r_2 \geq r_3 \geq r_4 \geq r_5$. Then $[123]$ is the greatest area among the ten triangles determined by these five points. Hence both A_4 and A_5 lie within the triangle $B_1 B_2 B_3$ which has A_1, A_2 and A_3 as the midpoints of its sides $B_2 B_3, B_3 B_1$ and $B_1 B_2$, respectively.

Suppose both A_4 and A_5 are inside $A_1 A_2 A_3$. Then

$$[124] + [234] + [314] = [125] + [235] + [315],$$

which implies that $r_4 = r_5$. Since A_4 and A_5 are on the same side of $A_1 A_2$ and of $A_2 A_3$, $A_4 A_5$ must be parallel to both segments. This is a contradiction since A_1, A_2 and A_3 are not collinear.

We may now assume by symmetry that A_4 is in $B_1 A_2 A_3$. Then $r_1 + r_4 = r_2 + r_3$. If A_5 is also in $B_1 A_2 A_3$, then $r_1 + r_5 = r_2 + r_3$ so that $r_4 = r_5$. This leads to a contradiction as before. On the other hand, if A_5 is in $A_2 A_3 B_2 B_3$, then $r_4 + r_5 = r_2 + r_3$ so that $r_1 = r_5$. This also leads to a contradiction.

Remark: The first solution is due to Bill Sands. The second solution is due to the proposer of the problem, Karel Horak, leader of the team from the Czech Republic.

4 Second Day, July 20, 1995

1. Find the maximum value of x_0 for which there exists a sequence of positive real numbers $x_0, x_1, \dots, x_{1995}$ satisfying the two conditions:

(a) $x_0 = x_{1995}$;

(b) $x_{i-1} + \frac{2}{x_{i-1}} = 2x_i + \frac{1}{x_i}$ for each $i = 1, \dots, 1995$.

Solution

The given condition is equivalent to $x_i^2 - \left(\frac{x_{i-1}}{2} + \frac{1}{x_{i-1}}\right)x_i + \frac{1}{2} = 0$, which yields either $x_i = \frac{1}{2}x_{i-1}$ or $x_i = \frac{1}{x_{i-1}}$.

We call the transition from x_{i-1} to x_i a move. Starting from x_0 , all possible moves are represented by arrows in Figure 1. The halving moves are represented by solid lines, while the reciprocating moves are represented by broken lines. We have to return to x_0 after exactly 1995 moves.

Note that each row consists of distinct numbers as long as $x_0 \neq 0$, and the two rows of numbers are either disjoint or identical. If they are disjoint, then it is not possible to return to x_0 after an odd number of moves, since each move is between a number enclosed by a circle and one by a square.

It follows that the two rows of numbers are identical. Even in this case, the task is only possible if the numbers in the first row enclosed by circles are identical to those in the second enclosed by squares. Moreover, the numbers in the two rows are descending in opposite direction. It follows that one of the numbers must be equal to 1, so that we can replace Figure 1 by Figure 2.

In order to maximize x_0 , there is no point in taking a reciprocal except on the very last move. Of the remaining 1994 moves, exactly half will be made on each side of 1. Hence the maximum value of x_0 is 2^{997} .

Alternative Solution

As in the first solution, either $x_i = \frac{x_{i-1}}{2}$ or $x_i = \frac{1}{x_{i-1}}$. For $i \geq 0$, we claim that $x_i = 2^{k_i} x_0^{\epsilon_i}$ for some integer k_i with $|k_i| \leq i$ and $\epsilon_i = (-1)^{k_i+i}$. This is true for $i = 0$, with $k_0 = 0$ and $\epsilon_0 = 1$, and we proceed by induction. If it is true for $i - 1$ and $x_i = \frac{1}{2} x_{i-1}$, then we have $k_i = k_{i-1} - 1$ and $\epsilon_i = \epsilon_{i-1}$; while if $x_i = \frac{1}{x_{i-1}}$, then we have $k_i = -k_{i-1}$ and $\epsilon_i = -\epsilon_{i-1}$. In each case, it is immediate that $|k_i| \leq i$ and $\epsilon_i = (-1)^{k_i+i}$. Thus $x_{1995} = 2^k x_0^\epsilon$, where $k = k_{1995}$ and $\epsilon = \epsilon_{1995}$, with $0 \leq |k| \leq 1995$ and $\epsilon = (-1)^{1995+k}$. It follows that $x_0 = x_{1995} = 2^k x_0^\epsilon$. If k is odd, then $\epsilon = 1$ and we have $2^k = 1$, a contradiction since $k \neq 0$. Thus k must be even, so that $\epsilon = -1$ and $x_0^2 = 2^k$. Since k is even and $|k| \leq 1995$, $k \leq 1994$. Hence $x_0 \leq 2^{997}$. We can have $x_0 = 2^{997}$, $x_i = \frac{1}{2} x_{i-1}$ for $i = 1, 2, \dots, 1994$, and $x_{1995} = \frac{1}{x_{1994}}$. Then

$$x_{1995} = \frac{1}{2^{-1994} x_0} = x_0$$

as desired.

Remark: The first solution is due to Johannes Notenboom, the leader of the team from the Netherlands. It is along the line of that of the proposer of the problem, Marcin Kuczma, the leader of the team from Poland, differing only in presentation. The second solution is due to Sam Maltby.

2. Let $ABCDEF$ be a convex hexagon with $AB = BC = CD$, $DE = EF = FA$, and $\angle BCD = \angle EFA = 60^\circ$.

Let G and H be two points in the interior of the hexagon such that $\angle AGB = \angle DHE = 120^\circ$. Prove that $AG + GB + GH + DH + HE \geq CF$.

Solution

Note that BCD and EFA are equilateral triangles. It follows that BE is an axis of symmetry of $ABDE$. Reflect BCD and EFA about BE to $BC'A$ and $EF'D$ respectively. Since $\angle BGA = 180^\circ - \angle AC'B$, G lies on the circumcircle of ABC' . Hence $\angle AGC' = \angle ABC' = 60^\circ$. Let K be the point on GC' such that KAG is equilateral. Then $\angle C'AK = 60^\circ - \angle BAK = \angle BAG$. Since $C'A = BA$ and $AK = AG$, triangles $C'AK$ and BAG are congruent. It follows that $GC' = GK + KC' = GA + GB$. Similarly, $DH + HE = HF'$. Hence

$$CF = C'F' \leq C'G + GH + HF' = AG + GB + GH + DH + HE,$$

with equality if and only if C', G, H and F' are collinear in that order.

Alternative Solution

The result holds without the condition that $\angle AGB = \angle DHE = 120^\circ$. Let C' and F' be as in the first solution. By Ptolemy's Inequality, $GC' \cdot AB \geq GA \cdot BC' + GB \cdot AC'$ so that $GC' \geq GA + GB$. Similarly, $HF' \geq HD + HE$, It follows that

$$CF = C'F' \leq C'G + GH + HF' \leq AG + GB + GH + HD + HE.$$

Remark: The first solution is due to Bill Sands who observes that the lemma $GC' = GA + GB$ is a special case of Ptolemy's Theorem which is featured in a 1973 Putnam Mathematics Competition. The second solution is due to Arthur Baragar. He uses Ptolemy's Inequality which is a stronger result than Ptolemy's Theorem.

3. Let p be an odd prime. Find the number of subsets A of $\{1, 2, \dots, 2p\}$ such that
- (a) A has exactly p elements, and
 - (b) the sum of all the elements in A is divisible by p .

Solution

For any p -element subset A of $\{1, 2, \dots, 2p\}$, denote by $s(A)$ the sum of the elements of A . Of the $\binom{2p}{p}$ such subsets, $B = \{1, 2, \dots, p\}$ and $C = \{p+1, p+2, \dots, 2p\}$ satisfy $s(B) = s(C) \equiv 0 \pmod{p}$. For $A \neq B, C$, we have $A \cap B \neq \emptyset \neq A \cap C$. Partition the $\binom{2p}{p} - 2$ p -element subsets other than B and C into groups of size p as follows. Two subsets A and A' are in the same group if and only if $A' \cap C = A \cap C$ and $A' \cap B$ is a cyclic permutation of $A \cap B$ within B . Suppose $A \cap B$ has n elements, $0 < n < p$. For some m such that $0 < m < p$,

$$A' \cap B = \{x + m : x \in A \cap B, x + m \leq p\}$$

$$\cup \{x + m - p : x \in A \cap B, x \leq p < x + m\}.$$

Hence $s(A') - s(A) \equiv mn \pmod{p}$, but mn is not divisible by p . It follows that exactly one subset A in each group satisfies $s(A) \equiv 0 \pmod{p}$, and the total number of such subsets is $p^{-1} \left(\binom{2p}{p} - 2 \right) + 2$.

Alternative Solution

Let ω be a primitive p -th root of unity. Then

$$\prod_{i=1}^{2p} (x - \omega^i) = (x^p - 1)^2 = x^{2p} - 2x^p + 1.$$

Comparing the coefficients of the term x^p , we have

$$2 = \sum \omega^{i_1+i_2+\dots+i_p} = \sum_{j=0}^{p-1} n_j \omega^j,$$

where the first summation ranges over all subsets $\{i_1, i_2, \dots, i_p\}$ of $\{1, 2, \dots, 2p\}$ and n_j in the second summation is the number of such subsets such that $i_1 + i_2 + \dots + i_p \equiv j \pmod{p}$. It follows that ω is a root of $G(x) = (n_0 - 2) + \sum_{j=1}^{p-1} n_j \omega^j$, which is a polynomial of degree $p - 1$. Since the minimal polynomial for ω over the field of rational numbers is $F(x) = \sum_{j=0}^{p-1} \omega^j$, which is also of degree $p - 1$, $G(x)$ must be a scalar multiple of $F(x)$, so that $n_0 - 2 = n_1 = n_2 = \dots = n_{p-1}$. Since $\sum_{j=0}^p n_j = \binom{2p}{p}$, we have $n_0 = p^{-1} \left(\binom{2p}{p} - 2 \right) + 2$.

Remark: The first solution is due to the proposer, Marcin Kuczma, the leader of the team from Poland. The second solution is due to Roberto Dvornicich, the leader of the team from Italy. Nikolay Nikolov, a Bulgarian student, won a special prize for his solution which is essentially along the line of the second one. Nikolay had won two Gold Medals and one Silver Medal at the last three IMO's, and topped off his outstanding career as a competitor by obtaining a perfect score this time.

37th International Mathematical Olympiad

Mumbai, India

Day I 9 a.m. - 1:30 p.m.

July 10, 1996

1. We are given a positive integer r and a rectangular board $ABCD$ with dimensions $|AB| = 20, |BC| = 12$. The rectangle is divided into a grid of 20×12 unit squares. The following moves are permitted on the board: one can move from one square to another only if the distance between the centers of the two squares is \sqrt{r} . The task is to find a sequence of moves leading from the square with A as a vertex to the square with B as a vertex.
 - (a) Show that the task cannot be done if r is divisible by 2 or 3.
 - (b) Prove that the task is possible when $r = 73$.
 - (c) Can the task be done when $r = 97$?
2. Let P be a point inside triangle ABC such that

$$\angle APB - \angle ACB = \angle APC - \angle ABC.$$

Let D, E be the incenters of triangles APB, APC , respectively. Show that AP, BD, CE meet at a point.

3. Let S denote the set of nonnegative integers. Find all functions f from S to itself such that

$$f(m + f(n)) = f(f(m)) + f(n) \quad \forall m, n \in S.$$

37th International Mathematical Olympiad

Mumbai, India

Day II 9 a.m. - 1:30 p.m.

July 11, 1996

1. The positive integers a and b are such that the numbers $15a + 16b$ and $16a - 15b$ are both squares of positive integers. What is the least possible value that can be taken on by the smaller of these two squares?
2. Let $ABCDEF$ be a convex hexagon such that AB is parallel to DE , BC is parallel to EF , and CD is parallel to FA . Let R_A, R_C, R_E denote the circumradii of triangles FAB, BCD, DEF , respectively, and let P denote the perimeter of the hexagon. Prove that

$$R_A + R_C + R_E \geq \frac{P}{2}.$$

3. Let p, q, n be three positive integers with $p + q < n$. Let (x_0, x_1, \dots, x_n) be an $(n + 1)$ -tuple of integers satisfying the following conditions:
 - (a) $x_0 = x_n = 0$.
 - (b) For each i with $1 \leq i \leq n$, either $x_i - x_{i-1} = p$ or $x_i - x_{i-1} = -q$.

Show that there exist indices $i < j$ with $(i, j) \neq (0, n)$, such that $x_i = x_j$.

39th International Mathematical Olympiad

Taipei, Taiwan

Day I

July 15, 1998

1. In the convex quadrilateral $ABCD$, the diagonals AC and BD are perpendicular and the opposite sides AB and DC are not parallel. Suppose that the point P , where the perpendicular bisectors of AB and DC meet, is inside $ABCD$. Prove that $ABCD$ is a cyclic quadrilateral if and only if the triangles ABP and CDP have equal areas.
2. In a competition, there are a contestants and b judges, where $b \geq 3$ is an odd integer. Each judge rates each contestant as either “pass” or “fail”. Suppose k is a number such that, for any two judges, their ratings coincide for at most k contestants. Prove that $k/a \geq (b - 1)/(2b)$.
3. For any positive integer n , let $d(n)$ denote the number of positive divisors of n (including 1 and n itself). Determine all positive integers k such that $d(n^2)/d(n) = k$ for some n .

39th International Mathematical Olympiad

Taipei, Taiwan

Day II

July 16, 1998

4. Determine all pairs (a, b) of positive integers such that $ab^2 + b + 7$ divides $a^2b + a + b$.
5. Let I be the incenter of triangle ABC . Let the incircle of ABC touch the sides BC , CA , and AB at K , L , and M , respectively. The line through B parallel to MK meets the lines LM and LK at R and S , respectively. Prove that angle RIS is acute.
6. Consider all functions f from the set N of all positive integers into itself satisfying $f(t^2 f(s)) = s(f(t))^2$ for all s and t in N . Determine the least possible value of $f(1998)$.

40th International Mathematical Olympiad

Bucharest

Day I

July 16, 1999

1. Determine all finite sets S of at least three points in the plane which satisfy the following condition:

for any two distinct points A and B in S , the perpendicular bisector of the line segment AB is an axis of symmetry for S .

2. Let n be a fixed integer, with $n \geq 2$.

(a) Determine the least constant C such that the inequality

$$\sum_{1 \leq i < j \leq n} x_i x_j (x_i^2 + x_j^2) \leq C \left(\sum_{1 \leq i \leq n} x_i \right)^4$$

holds for all real numbers $x_1, \dots, x_n \geq 0$.

(b) For this constant C , determine when equality holds.

3. Consider an $n \times n$ square board, where n is a fixed even positive integer. The board is divided into n^2 unit squares. We say that two different squares on the board are adjacent if they have a common side.

N unit squares on the board are marked in such a way that every square (marked or unmarked) on the board is adjacent to at least one marked square.

Determine the smallest possible value of N .

40th International Mathematical Olympiad

Bucharest

Day II

July 17, 1999

4. Determine all pairs (n, p) of positive integers such that

p is a prime,
 n not exceeded $2p$, and
 $(p - 1)^n + 1$ is divisible by n^{p-1} .

5. Two circles G_1 and G_2 are contained inside the circle G , and are tangent to G at the distinct points M and N , respectively. G_1 passes through the center of G_2 . The line passing through the two points of intersection of G_1 and G_2 meets G at A and B . The lines MA and MB meet G_1 at C and D , respectively.

Prove that CD is tangent to G_2 .

6. Determine all functions $f : \mathbf{R} \rightarrow \mathbf{R}$ such that

$$f(x - f(y)) = f(f(y)) + xf(y) + f(x) - 1$$

for all real numbers x, y .