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Notes in Math – Integration

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Definition

A) If $\frac{d}{dx}\left[f(x)\right] = g(x)$, then $f(x)$ is the primitive or integral of $g(x)$

$\frac{d}{dx}\left[f(x)\right] = 0 \Rightarrow f(x) = const$ or $f(x) = 0 \;\Rightarrow f(x) - const = 0$ in general $\therefore \int g(x)dx = f(x) - const$

B) where the constant can be fixed only after knowing the behavior of the function

Algebra of Integrals

Constant multipliers of function : $\int Kf(x)dx = K\int f(x)dx$

Constant multipliers of variable : $\int f(Kx)dx = \frac{1}{K}\int f(x)dx$

Seperation of Additive integrands : $\int [f \pm g]dx = \int f dx \pm \int g dx$

Replacing the variable : $\int f(x)dx = \int f(y)dy$

Substitution by another variable : $x = \phi(y) \Rightarrow \int f(x)dx = \int f(\phi(y))d\left[\phi(y)\right]$

Integral of a product : $\int [f.g]dx = f.\int g dx - \int \frac{df}{dx}\left[\int g dx\right]dx$

Integral of a ratio : $\int\left[\frac{f}{g}\right]dx = \int\left[f.\frac{1}{g}\right]dx$

Use F(*f*)LIATE rule to select *f* and *g* from the functions in the given problem. Since the *f*-function does not need integrating, use the most complicated one as *f* --- i.e. Function of a function, Logarithmic, Inverse, Algebraic, Trigonometric, Exponential

Standard Results

A) Algebraic

1) For $n \neq -1$, $\int x^n dx = \frac{x^{n+1}}{n+1} + c$

2) For $n = -1$, $\int \frac{1}{x} dx = \log(x) + c$

B) Trigonometric

1) $\int \sin(x)dx = -\cos(x) + c$

2) $\int \cos(x)dx = \sin(x) + c$

3) $\int \tan(x)dx = \log\left|\sec(x)\right| + c$

4) $\int \cot(x)dx = \log\left|\sin(x)\right| + c$

5) $\int \sec(x)dx = \log\left|\sec(x) + \tan(x)\right| + c$

6) $\int \text{cosec}(x)dx = \log\left|\text{cosec}(x) - \cot(x)\right| + c$

7) $\int \sec^2(x)dx = \tan(x) + c$

8) $\int \text{cosec}^2(x)dx = -\cot(x) + c$

9) $\int \sec(x)\tan(x)dx = \sec(x) + c$

10) $\int \operatorname{cosec}(x)\cot(x)\,dx = -\operatorname{cosec}(x) + c$

C) Exponential

1) $\int e^x\,dx = e^x + c$

2) $\int a^x\,dx = \dfrac{a^x}{\log(a)} + c,\ a > 0$

D) By First Principles

1) $\int \dfrac{f'}{f}\,dx = \log|f| + c$

2) $\int e^x\left[f + f'\right]dx = e^x f + c$

E) Trigonometric functions as variables
Write the variable as some basic trigonometric function s.t. through use of identities, the integrand simplifies to a simple algebraic expression.

1) $\int \dfrac{1}{\sqrt{a^2 - x^2}}\,dx = \sin^{-1}\left(\dfrac{x}{a}\right) + c;$ substituting $x = a\sin(\theta)$

2) $\int \dfrac{1}{a^2 + x^2}\,dx = \dfrac{1}{a}\tan^{-1}\left(\dfrac{x}{a}\right) + c;$ substituting $x = a\tan(\theta)$

3) $\int \dfrac{1}{a^2 - x^2}\,dx = \dfrac{1}{2a}\log\left(\dfrac{a+x}{a-x}\right) + c;$ substituting $x = a\sin(\theta)$

4) $\int \dfrac{1}{x^2 - a^2}\,dx = \dfrac{1}{2a}\log\left(\dfrac{x-a}{x+a}\right) + c;$ substituting $x = a\operatorname{cosec}(\theta)$

5) $\int \dfrac{1}{x\sqrt{x^2 - a^2}}\,dx = \dfrac{1}{a}\sec^{-1}\left(\dfrac{x}{a}\right) + c;$ substituting $x = a\sec(\theta)$

6) $\int \dfrac{1}{\sqrt{x^2 \pm a^2}}\,dx = \log\left(\dfrac{x + \sqrt{x^2 \pm a^2}}{a}\right) + c;$ substituting $x = a\tan(\theta)$

7) $\int \sqrt{a^2 - x^2}\,dx = \dfrac{x}{2}\sqrt{a^2 - x^2} + \dfrac{a^2}{2}\sin^{-1}\left(\dfrac{x}{a}\right) + c,$ by substituting $x = a\sin\theta$

8) $\int \sqrt{x^2 - a^2}\,dx = \dfrac{x}{2}\sqrt{x^2 - a^2} + \dfrac{a^2}{2}\cosh^{-1}\left(\dfrac{x}{a}\right) + c,$ by substituting $x = a\sec\theta$

9) $\int \sqrt{a^2 + x^2}\,dx = \dfrac{x}{2}\sqrt{a^2 + x^2} + \dfrac{a^2}{2}\sinh^{-1}\left(\dfrac{x}{a}\right) + c,$ by substituting $x = a\tan\theta$

F) Expressions with powers of T-functions (Reduction Formulae)

1) $\int \sin^2(ax)\,dx = \dfrac{x}{2} - \dfrac{\sin(2ax)}{4a} + c$

2) $\int \cos^2(ax)\,dx = \dfrac{x}{2} + \dfrac{\sin(2ax)}{4a} + c$

3) $\int \sin^3(ax)\,dx = -\dfrac{1}{a}\cos(ax) - \dfrac{1}{3a}\cos^3(ax) + c$

4) $\int \cos^3(ax)\,dx = \dfrac{1}{a}\sin(ax) - \dfrac{1}{3a}\sin^3(ax) + c$

5) $\int \sin^n(ax)\,dx = -\dfrac{\sin^{n-1}(ax)\cos(ax)}{na} + \dfrac{n-1}{n}\int \sin^{n-2}(ax)\,dx + c$

6) $\int \cos^n(ax)\,dx = \dfrac{\cos^{n-1}(ax)\sin(ax)}{na} + \dfrac{n-1}{n}\int \cos^{n-2}(ax)\,dx + c$

7) $\int \dfrac{dx}{1 \pm \sin(ax)} = \mp \tan\left[\dfrac{\pi}{4} \mp \dfrac{ax}{2}\right] + c$

G) Inverse T-functions

1) $\int \sin^{-1}(ax)\,dx = x\sin^{-1}(ax) + \dfrac{1}{a}\sqrt{1-a^2x^2} + c$

2) $\int \cos^{-1}(ax)\,dx = x\cos^{-1}(ax) - \dfrac{1}{a}\sqrt{1-a^2x^2} + c$

H) Algebraic & Trigonometric

1) $\int x\sin(ax)\,dx = \dfrac{1}{a^2}\sin(ax) - \dfrac{1}{a}x\cos(ax) + c$

2) $\int x\cos(ax)\,dx = \dfrac{1}{a^2}\cos(ax) + \dfrac{1}{a}x\sin(ax) + c$

3) $\int \sin(ax)\cos(bx)\,dx = -\dfrac{\cos(a-b)x}{2(a-b)} - \dfrac{\cos(a+b)x}{2(a+b)} + c$

4) $\int xe^{ax}\,dx = \dfrac{e^{ax}}{a^2}(ax-1) + c$

5) $\int xb^{ax}\,dx = \dfrac{xb^{ax}}{a\ln(b)} - \dfrac{b^{ax}}{a^2(\ln(b))^2} + c$

6) $\int e^{ax}\sin(bx)\,dx = \dfrac{e^{ax}}{a^2+b^2}\left(a\sin(bx) - b\cos(bx)\right) + c$

7) $\int e^{ax}\cos(bx)\,dx = \dfrac{e^{ax}}{a^2+b^2}\left(a\cos(bx) + b\sin(bx)\right) + c$

8) $\int x^n\ln(ax)\,dx = x^{n+1}\left(\dfrac{\ln(ax)}{x+1} - \dfrac{1}{(n+1)^2}\right) + c$

I) Quadratic expressions

1) $$\int \frac{dx}{ax^2+bx+c} = \begin{cases} \dfrac{1}{\sqrt{\Delta}}\ln\left|\dfrac{2ax+b-\sqrt{\Delta}}{2ax+b+\sqrt{\Delta}}\right| + c & \Delta > 0 \\ \dfrac{1}{2ax+b} + c & \Delta = 0 \quad \text{where } \Delta = b^2 - 4ac \\ \dfrac{2}{\sqrt{-\Delta}}\tan^{-1}\left(\dfrac{2ax+b}{\sqrt{-\Delta}}\right) + c & \Delta < 0 \end{cases}$$

2) $$\int \frac{dx}{\sqrt{ax^2+bx+c}} = \begin{cases} \dfrac{1}{\sqrt{a}}\ln\left|\dfrac{2ax+b+2\sqrt{a}}{\sqrt{ax^2+bx+c}}\right| + c & a > 0 \\ \dfrac{1}{\sqrt{-a}}\sin^{-1}\left(\dfrac{-2ax-b}{\sqrt{b^2-4ac}}\right) + c & a < 0 \end{cases}$$

Standard Techniques

Integrating a given expression is usually a job of reducing the expression to a standard form whose integral is already derived from first principles. To do so one can use various techniques:-

Function contains	***Technique / Trick***
Surds	Rationalize
Surds: $(ax+b)^{1-n}, n>0$	Substitute $z=(ax+b)\frac{1}{n}$
Surds: $\frac{\sqrt{ax+b}}{\sqrt{cx+d}}$	Substitute $z=\sqrt{cx+d}$
Polynomial factors	Expand terms
Single Trigonometric functions	Convert to $\frac{1}{f}\frac{df}{dx}$ form by Multiply and divide by a sum of T-functions
Different T-functions	Use identities to obtain single function
Different arguments for T-functions	Use compound angle formulae to get into similar angle
Products of T-functions	Try expressing one as a derivative of the other / substitute…
Ratio of polynomials	Express denominator as a perfect square / product of factors or express numerator as derivative of denominator
$\frac{af+bg}{cf+dg}$, where $f'=Kg$	Write $N = AD + BD'$. To obtain A & B, equate the coefficients of similar functions to get two linear equations. $I = Ax + B\log\lvert f\rvert + c$
$\frac{1}{a\sin x+b\cos x}$, where a,b$\in R$	Express $a=r\cos\theta, b=r\sin\theta \Rightarrow a^2+b^2=r^2, \tan\theta=\frac{b}{a}$
Even powers of Sin, Cos	Use $\sin^2 x=\frac{1-\cos 2x}{2}, \cos^2 x=\frac{1+\cos 2x}{2}$ to reduce
Odd powers of Sin, Cos	$\int T^n dx=\int T^{n-1}Tdx$, T^{n-1} is now even power T-function, while Tdx forms the differential when substitution is used
Powers of Tan, Cot OR Even powers of Sec, Cosec	Seperate $\tan^2/\cot^2/\sec^2/\mathrm{cosec}^2$: $\int\tan^n xdx=\int\tan^{n-2}x\left(\sec^2 x-1\right)dx$ $\sec^2 x$ forms the differential after substitution. Keep reducing.
$\frac{1}{ax^2+bx+c}$	Divide by a and then complete the square
$\frac{px+q}{ax^2+bx+c}$	Adjust the numerator to be like $\frac{d}{dx}$[Denominator]
$\frac{N^m(x)}{D^n(x)}, m>n$	Reduce the degree of the numerator by division/ factorization and then use the above methods
$D(x)$ has only linear factors	Method of Partial Fractions Express $N(x)=AD_1(x)+BD_2(x)+CD_3(x)+\ldots$, where $D_1, D_2, D_3, \ldots$ are linear factors of the denominator. Open brackets and simplify RHS. Obtain constants A and B by equating the coefficients of the like terms of LHS and RHS.
$D(x)$ has one quadratic factors	Write $N(x)=AD_1(x)+(Bx+C)D_2(x)$, where D_1 is the quadratic factor. Then solve as per the method of partial fractions
$D(x)$ has only non-linear factors	Use substitution to get $D(x)$ to have linear factors and then use Partial Fractions